A NOTE ON THE METHODS OF CONSTRUCTION OF SOME BALANCED GENERALIZED TWO-WAY ELIMINATION OF HETEROGENEITY DESIGNS*

BY

G. NAGESWARA RAO

Punjab Agricultural University, Ludhiana

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1. INTRODUCTION

The general method of construction of arrangement of v symbols in a $v \times v$ square such that (*a*) diagonal cells are blank, (*b*) every symbol occurs at most once in any row or column and (*c*) the i-th symbol does not occur in the i-th row or the i-th column ($i=1, 2, \ldots, v$), is not known, for even v [*cf.* Raghavarao (1971), pp. 110]. Agrawal (1966) gave the method of construction of this class of designs for v odd and trial-and-error solutions for $v=4$ and 16. Raghavarao (1970) proved the theorem of existence of construction of this class of designs for any v alongwith trial-and-error solutions for $v=6$, 8 and 10. In this note, we present a general method of construction of this class of designs when v is even and different from 2.

2. METHODS OF CONSTRUCTION

When v is even, we distinguish two cases. (*i*) $v=4t$ (*ii*) $v=4t+2$, t being a positive integer.

Case (*i*) $v=4t$.

Let A be the arrangement of first $2t$ symbols in cyclic order in a $2t \times 2t$ square and let B be the arrangement of remaining $2t$ symbols *i.e.* from $2t+1$ to $4t$ in cyclic order in a $2t \times 2t$ square. Write

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AB in a cyclic order for all the $4t$ symbols as given in (2.1):

$$\begin{matrix} A & B \\ B & A \end{matrix} \quad \ldots(2.1)$$

In (2.1), interchange the symbols $(2t+2)$ and 2 in both 2nd and $(2t+2)$nd rows, symbols $(2t+4)$ and 4 in both 3rd and $(2t+3)$rd rows, and so on till symbols $(4t-2)$ and $2(t-1)$ in both t-th and $3t$-th rows. Similarly, interchange the symbols $(2t+3)$ and 3 in both 2nd and $(2t+2)$nd rows, symbols $(2t+5)$ and 5 in both 3rd and $(2t+3)$rd rows, and so on till symbols $(4t-1)$ and $(2t-1)$ in both, t-th and $3t$-th rows. Let us denote the columns of (2.1) by $\alpha_1, \alpha_2, \ldots, \alpha_{2t}, \alpha_{2t+1}, \alpha_{2t+2}, \ldots, \alpha_{4t}$ after the interchanging of symbols is effected. Now, rearrange the columns of (2.1) as given in 2.2)

$$\alpha_1, \alpha_{2t+1}, \alpha_{2t+2}, \ldots, \alpha_{4t}, \alpha_2, \alpha_3, \ldots, \alpha_{2t} \quad \ldots(2.2)$$

In (2.2), we observe that each of the $4t$ symbols occurs exactly once in the diagonal positions. Now, mapping the elements of the diagonal positions on 1, 2,..., $4t$ in (2.2) and omitting the diagonal positions, we get the required arrangement.

Case (*ii*) $v=4t+2$, $t>0$

The arrangemeut cannot be constructed when $t=0$, and theorem 6.5.7 of Rghavarao (1971, PP. 109) holds for $t>0$.

Let C be the arrangement of first $(2t+1)$ symbols in cyclie order in a $(2t+1)\times(2t+1)$ square and let D be the arrangement of remaining $(2t+1)$ symbols *i.e.* from $(2t+2)$ to $(4t+2)$ in cyclic order in a $(2t+1)\times(2t+1)$ square. Then, write CD in a cyclic order for all the $(4t+2)$ symbols as given in (2.3)

$$\begin{matrix} C & D \\ D & C \end{matrix} \quad \ldots(2.3)$$

In (2.3), interchange the symbols $(2t+3)$ and 2 in both 2nd and $(2t+3)$rd rows. symbols $(2t+5)$ and 4 in both 3rd and $(2t+4)$-th rows, and so on up to symbols $(4t+1)$ and $2t$ in both $(t+1)$-th and $(3t+2)$-th rows. Similarly, interchange the symbols $(2t+4)$ and 3 in both 2nd and $(2t+3)$ rd rows, symbols $(2t+6)$ and 5 in both 3rd and $(2t+4)$-th rows, and so on upto symbols $4t$ and $(2t-1)$ in both t-th and $(3t+1)$st rows. Let us denote the columns of (2.3) by $\beta_1, \beta_2, \ldots,$

$\beta_{2t+1}, \beta_{2t+2}, \ldots, \beta_{4t+2}$ after the interchanging of the above symbols is effected. Now, rearrange the columns of (2.3) as given in (2.4)

$$\beta_1, \beta_{2t+2}, \beta_{2t+3}, \ldots, \beta_{4t+2}, \beta_2, \beta_3, \ldots \beta_{2t+1} \qquad \ldots(2.4)$$

In (2.4), we observe that each of the $(4t+2)$ symbols occurs exactly once in the diagonal positions. Now, mapping the elements of the diagonal positions on $1, 2, \ldots, 4t+2$ in (2.4) and omitting the diagonal positions, we get the required arrangement.

3. Illustrative Examples

In this section, we illustrate the above methods of construction with the help of two examples, one for each of the two cases.

Case (*i*) We construct the design for $v=12$.

$$\text{Let } A = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{matrix} \text{ and } B = \begin{matrix} 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 9 & 10 & 11 & 12 & 7 \\ 9 & 10 & 11 & 12 & 7 & 8 \\ 10 & 11 & 12 & 7 & 8 & 9 \\ 11 & 12 & 7 & 8 & 9 & 10 \\ 12 & 7 & 8 & 9 & 10 & 11 \end{matrix}$$

Then, write AB in cyclic order as in (2.1)

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \underline{2} & \underline{3} & 4 & 5 & 6 & 1 & \underline{8} & \underline{9} & 10 & 11 & 12 & 7 \\ 3 & \underline{4} & \underline{5} & 6 & 1 & 2 & 9 & \underline{10} & \underline{11} & 12 & 7 & 8 \\ 4 & 5 & 6 & 1 & 2 & 3 & 10 & 11 & 12 & 7 & 8 & 9 \\ 5 & 6 & 1 & 2 & 3 & 4 & 11 & 12 & 7 & 8 & 9 & 10 \\ 6 & 1 & 2 & 3 & 4 & 5 & 12 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\ \underline{8} & \underline{9} & 10 & 11 & 12 & 7 & \underline{2} & \underline{3} & 4 & 5 & 6 & 1 \\ 9 & \underline{10} & \underline{11} & 12 & 7 & 8 & 3 & \underline{4} & \underline{5} & 6 & 1 & 2 \\ 10 & 11 & 12 & 7 & 8 & 9 & 4 & 5 & 6 & 1 & 2 & 3 \\ 11 & 12 & 7 & 8 & 9 & 10 & 5 & 6 & 1 & 2 & 3 & 4 \\ 12 & 7 & 8 & 9 & 10 & 11 & 6 & 1 & 2 & 3 & 4 & 5 \end{matrix} \qquad \ldots(3.1)$$

In (3.1), interchange the symbols 8 and 2 in both 2nd and 8-th rows and symbols 10 and 4 in both 3rd and 9-th rows. Similarly, interchange the symbols 9 and 3 in both 2nd and 8-th rows and 11 and 5 in both 3rd and 9th rows. The symbols to be interchanged in each row are underlined in (3.1). Now rearrange the columns after interchanging of the above mentioned symbols is effected, as given in (2.2). After rearranging of columns, the new arrangement is given in (3.2).

1	7	8	9	10	11	12	2	3	4	5	6
8	2	3	10	11	12	7	9	4	5	6	1
3	9	4	5	12	7	8	10	11	6	1	2
4	10	11	12	7	8	9	5	6	1	2	3
5	11	12	7	8	9	10	6	1	2	3	4
6	12	7	8	9	10	11	1	2	3	4	5
7	1	2	3	4	5	6	8	9	10	11	12
2	8	9	4	5	6	1	3	10	11	12	7
9	3	10	11	6	1	2	4	5	12	7	8
10	4	5	6	1	2	3	11	12	7	8	9
11	5	6	1	2	3	4	12	7	8	9	10
12	6	1	2	3	4	5	7	8	9	10	11

(3.2)

In (3.2), we observe that each of the 12 symbols occurs exactly once in the diagonal positions. Now, mapping the elements of the diagonal positions on 1, 2,..., 12 in (3.2) and omiting the diagonal positions, we arrive at the required arrangement as given in (3.3).

x	10	5	11	6	12	4	2	8	3	9	7
5	x	8	6	12	4	10	11	3	9	7	1
8	11	x	9	4	10	5	6	12	7	1	2
3	6	12	x	10	5	11	9	7	1	2	8
9	12	4	10	x	11	6	7	1	2	8	3
7	4	10	5	11	x	12	1	2	8	3	9
10	1	2	8	3	9	x	5	11	6	12	4
2	5	11	3	9	7	1	x	6	12	4	10
11	8	6	12	7	1	2	3	x	4	10	5
6	3	9	7	1	2	8	12	4	x	5	11
12	9	7	1	2	8	3	4	10	5	x	6
4	7	1	2	8	3	9	10	5	11	6	x

(3.3)

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Case (*ii*) We construct the design for $v=22$.

Let C=

1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	1
3	4	5	6	7	8	9	10	11	1	2
4	5	6	7	8	9	10	11	1	2	3
5	6	7	8	9	10	11	1	2	3	4
6	7	8	9	10	11	1	2	3	4	5
7	8	9	10	11	1	2	3	4	5	6
8	9	10	11	1	2	3	4	5	6	7
9	10	11	1	2	3	4	5	6	7	8
10	11	1	2	3	4	5	6	7	8	9
11	1	2	3	4	5	6	7	8	9	10

and D=

12	13	14	15	16	17	18	19	20	21	22
13	14	15	16	17	18	19	20	21	22	12
14	15	16	17	18	19	20	21	22	12	13
15	16	17	18	19	20	21	22	12	13	14
16	17	18	19	20	21	22	12	13	14	15
17	18	19	20	21	22	12	13	14	15	16
18	19	20	21	22	12	13	14	15	16	17
19	20	21	22	12	13	14	15	16	17	18
20	21	22	12	13	14	15	16	17	18	19
21	22	12	13	14	15	16	17	18	19	20
22	12	13	14	15	16	17	18	19	20	21

Then, write CD in cyclic order as in (2.3)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
2	**3**	4	5	6	7	8	9	10	11	1	**13**	**14**	15	16	17	18	19	20	21	22	12
3	**4**	**5**	6	7	8	9	10	11	1	2	14	**15**	**16**	17	18	19	20	21	22	12	13
4	5	**6**	**7**	8	9	10	11	1	2	3	15	16	**17**	**18**	19	20	21	22	12	13	14
5	6	7	**8**	**9**	10	11	1	2	3	4	16	17	18	**19**	**20**	21	22	12	13	14	15
6	7	8	9	**10**	11	1	2	3	4	5	17	18	19	20	**21**	22	12	13	14	15	16
7	8	9	10	11	1	2	3	4	5	6	18	19	20	21	22	12	13	14	15	16	17
8	9	10	11	1	2	3	4	5	6	7	19	20	21	22	12	13	14	15	16	17	18
9	10	11	1	2	3	4	5	6	7	8	20	21	22	12	13	14	15	16	17	18	19
10	11	1	2	3	4	5	6	7	8	9	21	22	12	13	14	15	16	17	18	19	20
11	1	2	3	4	5	6	7	8	9	10	22	12	13	14	15	16	17	18	19	20	21 (3.4)
12	13	14	15	16	17	18	19	20	21	22	1	2	3	4	5	6	7	8	9	10	11
13	**14**	15	16	17	18	19	20	21	22	12	**2**	**3**	4	5	6	7	8	9	10	11	1
14	**15**	**16**	17	18	19	20	21	22	12	13	3	**4**	**5**	6	7	8	9	10	11	1	2
15	16	**17**	**18**	19	20	21	22	12	13	14	4	5	**6**	**7**	8	9	10	11	1	2	3
16	17	18	**19**	**20**	21	22	12	13	14	15	5	6	7	**8**	**9**	10	11	1	2	3	4
17	18	19	20	**21**	22	12	13	14	15	16	6	7	8	9	**10**	11	1	2	3	4	5
18	19	20	21	22	12	13	14	15	16	17	7	8	9	10	11	1	2	3	4	5	6
19	20	21	22	12	13	14	15	16	17	18	8	9	10	11	1	2	3	4	5	6	7
20	21	22	12	13	14	15	16	17	18	19	9	10	11	1	2	3	4	5	6	7	8
21	22	12	13	14	15	16	17	18	19	20	10	11	1	2	3	4	5	6	7	8	9
22	12	13	14	15	16	17	18	19	20	21	11	1	2	3	4	5	6	7	8	9	10

In (3.4), interchange the symbols 13 and 2 in both 2nd and 13-th rows, symbols 15 and 4 in both 3rd and 14-th rows, symbols 17 and 6 in both 4-th and 15-th rows, symbols 19 and 8 in both 5-th and 16-th rows and symbols 21 and 10 in both 6-th and 17-th rows. Similarly, interchange the symbols 14 and 3 in both 2nd and 13-th rows, symbols 16 and 5 in both 3rd and 14-th rows, symbols 18 and 7 in both 4-th and 15-th rows and symbols 20 and 9 in both 5-th and 16-th rows. The symbols to be interchaged in each row are underlined in (3.4). Now, rearrange the columns after interchanging of the above symbols is effected, as given in (2.4). After rearranging the columns, the new arrangement is given in (3.5).

1	12	13	14	15	16	17	18	19	20	21	22	2	3	4	5	6	7	8	9	10	11
13	2	3	15	16	17	18	19	20	21	22	12	14	4	5	6	7	8	9	10	11	1
3	14	4	5	17	18	19	20	21	22	12	13	15	16	6	7	8	9	10	11	1	2
4	15	16	6	7	19	20	21	22	12	13	14	5	17	18	8	9	10	11	1	2	3
5	16	17	18	8	9	21	22	12	13	14	15	6	7	19	20	10	11	1	2	3	4
6	17	18	19	20	10	22	12	13	14	15	16	7	8	9	21	11	1	2	3	4	5
7	18	19	20	21	22	12	13	14	15	16	17	8	9	10	11	1	2	3	4	5	6
8	19	20	21	22	12	13	14	15	16	17	18	9	10	11	1	2	3	4	5	6	7
9	20	21	22	12	13	14	15	16	17	18	19	10	11	1	2	3	4	5	6	7	8
10	21	22	12	13	14	15	16	17	18	19	20	11	1	2	3	4	5	6	7	8	9
11	22	12	13	14	15	16	17	18	19	20	21	1	2	3	4	5	6	7	8	9	10
12	1	2	3	4	5	6	7	8	9	10	11	13	14	15	16	17	18	19	20	21	22
2	13	14	4	5	6	7	8	9	10	11	1	3	15	16	17	18	19	20	21	22	12
14	3	15	16	6	7	8	9	10	11	1	2	4	5	17	18	19	20	21	22	12	13
15	4	5	17	18	8	9	10	11	1	2	3	16	6	7	19	20	21	22	12	13	14
16	5	6	7	19	20	10	11	1	2	3	4	17	18	8	9	21	22	12	13	14	15
17	6	7	8	9	21	11	1	2	3	4	5	18	19	20	10	22	12	13	14	15	16
18	7	8	9	10	11	1	2	3	4	5	6	19	20	21	22	12	13	14	15	16	17
19	8	9	10	11	1	2	3	4	5	6	7	20	21	22	12	13	14	15	16	17	18
20	9	10	11	1	2	3	4	5	6	7	8	21	22	12	13	14	15	16	17	18	19
21	10	11	1	2	3	4	5	6	7	8	9	22	12	13	14	15	16	17	18	19	20
22	11	1	2	3	4	5	6	7	8	9	10	12	13	14	15	16	17	18	19	20	21

(3.5)

In (3·5), we observe that each of the 22 symbols occur exactly once in the diagonal positions. Now mapping the elements of the diagonal positions on 1, 2, ..., 22 in (3.5) and omitting the diagonal positions, we arrive at the required arrangement as given in (3.6).

x	7	18	8	19	9	20	10	21	11	22	17	2	13	3	14	4	15	5	16	6	12
18	x	13	19	9	20	10	21	11	22	17	7	8	3	14	4	15	5	16	6	12	1
13	8	x	14	20	10	21	11	22	17	7	18	19	9	4	15	5	16	6	12	1	2
3	19	9	x	15	21	11	22	17	7	18	8	14	20	10	5	16	6	12	1	2	13
14	9	20	10	x	16	22	17	7	18	8	19	4	15	21	11	6	12	1	2	13	3
4	20	10	21	11	x	17	7	18	8	19	9	15	5	16	22	12	1	2	13	3	14
15	10	21	11	22	17	x	18	8	19	9	20	5	16	6	12	1	2	13	3	14	4
5	21	11	22	17	7	18	x	19	9	20	10	16	6	12	1	2	13	3	14	4	15
16	11	22	17	7	18	8	19	x	20	10	21	6	12	1	2	13	3	14	4	15	5
6	22	17	7	18	8	19	9	20	x	21	11	12	1	2	13	3	14	4	15	5	16
12	17	7	18	8	19	9	20	10	21	x	22	1	2	13	3	14	4	15	5	16	6
7	1	2	13	3	14	4	15	5	16	6	x	18	8	19	9	20	10	21	11	22	17
2	18	8	3	14	4	15	5	16	6	12	1	x	19	9	20	10	21	11	22	17	7
8	13	19	9	4	15	5	16	6	12	1	2	3	x	20	10	21	11	22	17	7	18
19	3	14	20	10	5	16	6	12	1	2	13	9	4	x	21	11	22	17	7	18	8
9	14	4	15	21	11	6	12	1	2	13	3	20	10	5	x	22	17	7	18	8	19
20	4	15	5	16	22	12	1	2	13	3	14	10	21	11	6	x	7	18	8	19	9
10	15	5	16	6	12	1	2	13	3	14	4	21	11	22	17	7	x	8	19	9	20
21	5	16	6	12	1	2	13	3	14	4	15	11	22	17	7	18	8	x	9	20	10
11	16	6	12	1	2	13	3	14	4	15	5	22	17	7	18	8	19	9	x	10	21
22	6	12	1	2	13	3	14	4	15	5	16	17	7	18	8	19	9	20	10	x	11
17	12	1	2	13	3	14	4	15	5	16	6	7	18	8	19	9	20	10	21	11	x

(3.6)